Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Journal of Sound and Vibration 285 (2005) 1235–1241
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Short Communication

# A convolution integral method for certain strongly nonlinear oscillators

H. Hu*, J.H. Tang

*School of Civil Engineering, Hunan University of Science and Technology, Xiangtan 411201, Hunan, PR China*

Received 5 May 2004; received in revised form 18 June 2004; accepted 11 November 2004
Available online 20 January 2005

**Abstract**

A modification of the convolution integral method for linear oscillators is presented for the analysis of certain strongly nonlinear oscillators. The modification provides an iteration scheme. Two examples are given to illustrate the effectiveness of the proposed method.
© 2004 Elsevier Ltd. All rights reserved.

## 1. Introduction

Many perturbation techniques exist for constructing analytical approximations to the oscillatory solution of second-order, nonlinear differential equations [1,2]. But many of them apply to weakly nonlinear cases only. To overcome the limitations, many novel techniques have been proposed in recent years. For example, Cheung et al. [3] proposed a modified Lindstedt–Poincare method, and Lim et al. [4] presented a modified Mickens procedures for certain nonlinear oscillators. Recently, He [5] proposed a perturbation technique which is valid for the Duffing equation with large parameters.

Consider a nonlinear oscillator modeled by

$$\ddot{x} + F(x) = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{1}$$

*Corresponding author.
*E-mail address:* huihuxt@yahoo.com.cn (H. Hu).

0022-460X/$ - see front matter © 2004 Elsevier Ltd. All rights reserved.
doi:10.1016/j.jsv.2004.11.023

where overdots denote differentiation with respect to time $t$ and $F(x)$ satisfies the condition

$$F(-x) = -F(x). \tag{2}$$

In this paper, a new approximate method is presented for studying the nonlinear oscillator modeled by Eqs. (1) and (2), in which the convolution integral is used. The convolution integral for linear oscillators is stated briefly in Section 2. In Section 3, the convolution integral for linear oscillators is then used for nonlinear oscillators. In Section 4, two examples are given to illustrate the effectiveness of the proposed method.

## 2. The convolution integral for linear oscillators

Consider a single degree of freedom system described by the linear differential equation of motion

$$m\ddot{x} + kx = F(t), \quad x(0) = A, \quad \dot{x}(0) = 0. \tag{3}$$

This equation can be rewritten as

$$\ddot{x} + \omega_n^2 x = f(t), \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{4}$$

where $f(t) = F(t)/m$ and $\omega_n^2 = k/m$ is the natural frequency of linear system (3). Using Eq. (2.120) in Ref. [6] and letting $\zeta = 0$ in Eq. (2.120), the solution to Eq. (4) can be written in the form

$$x(t) = A\cos\omega_n t + \int_0^t \frac{f(\tau)}{\omega_n}\sin\omega_n(t-\tau)\,\mathrm{d}\tau. \tag{5}$$

The integral

$$\int_0^t \frac{f(\tau)}{\omega_n}\sin\omega_n(t-\tau)\,\mathrm{d}\tau$$

is called the convolution or Duhamel integral [7]. It should be pointed out that this integral can also obtained by using the Green's function approach.

## 3. The convolution integral for nonlinear oscillators

Supposing that the natural frequency of Eq. (1) is $\omega$, which is unknown to be further determined, we can obtain the linearized Eq. (1), which reads [5]

$$\ddot{x} + \omega^2 x = 0, \quad \dot{x}(0) = A, \quad \dot{x}(0) = 0.$$

It follows from this equation and Eq. (1) that $f(x) = \omega^2 x - F(x)$ may be "small" [8]. To apply the convolution integral to nonlinear oscillators, we then rewrite Eq. (1) to read [4,5,8]

$$\ddot{x} + \omega^2 x = \omega^2 x - F(x) =: f(x(t)), \quad x(0) = A, \quad \dot{x}(0) = 0. \tag{6}$$

Comparing Eq. (6) with Eq. (4) and using Eq. (5), we get the following exact "solution" to Eq. (1):

$$x(t) = A\cos\omega t + \frac{1}{\omega}\int_0^t [\omega^2 x(\tau) - F(x(\tau))]\sin\omega(t-\tau)\,\mathrm{d}\tau. \tag{7}$$

To obtain the approximate analytical solution, we can substitute $x(t) \approx A\cos\omega t$ into the right-hand side of Eq. (7). Therefore, Eq. (7) becomes

$$x(t) \approx A\cos\omega t + \frac{1}{\omega}\int_0^t [\omega^2 A\cos\omega\tau - F(A\cos\omega\tau)]\sin\omega(t-\tau)\,\mathrm{d}\tau. \tag{8}$$

The initial details of this method will be illustrated by applying it to two examples.

## 4. Examples

**Example 1.** Consider the Duffing equation

$$\ddot{x} + x + \varepsilon x^3 = 0, \quad x(0) = A, \quad \dot{x}(0) = 0. \tag{9}$$

For this example, $F(x) = x + \varepsilon x^3$. Eq. (8) gives

$$\begin{aligned} x(t) &\approx A\cos\omega t + \frac{1}{\omega}\int_0^t [(\omega^2 - 1)A\cos\omega\tau - \varepsilon A^3\cos^3\omega\tau]\sin\omega(t-\tau)\,\mathrm{d}\tau \\ &= A\cos\omega t + \frac{1}{\omega}\int_0^t [(\omega^2 - 1 - 3\varepsilon A^2/4)A\cos\tau - \frac{\varepsilon A^3}{4}\cos 3\omega\tau]\sin\omega(t-\tau)\,\mathrm{d}\tau \\ &= A\cos\omega t + \frac{A}{2\omega}(\omega^2 - 1 - 3\varepsilon A^2/4)t\sin\omega t + \frac{\varepsilon A^3}{32\omega^2}(\cos 3\omega t - \cos\omega t). \end{aligned} \tag{10}$$

No secular term requires

$$\omega = \sqrt{1 + 3\varepsilon A^2/4}. \tag{11}$$

Therefore, the approximate solution to Eq. (9) is

$$x(t) = A\cos\omega t + \frac{\varepsilon A^3}{32\omega^2}(\cos 3\omega t - \cos\omega t). \tag{12}$$

Eq. (11) is just the standard harmonic balance result [1] and Eq. (12) in agreement with the result in Refs. [9,10]. The discrepancy of the approximate solution given by Eq. (11) with respect to the exact solution is less than 2.22% [4]. It has been shown that Eq. (12) can give good approximations for very large values of $\varepsilon A^2$ [11].

**Example 2.** Consider the nonlinear differential equation [12,13]

$$\ddot{x} + x^{1/3} = 0, \quad x(0) = A, \quad \dot{x}(0) = 0. \tag{13}$$

For this example, $F(x) = x^{1/3}$, and Eq. (8) results in

$$x(t) \approx A\cos\omega t + \frac{1}{\omega}\int_0^t [\omega^2 A\cos\omega\tau - A^{1/3}(\cos\omega\tau)^{1/3}]\sin\omega(t-\tau)\,\mathrm{d}\tau. \tag{14}$$

It can be shown that

$$(\cos\theta)^{1/3} = b_1\cos\theta + b_3\cos 3\theta + \cdots, \tag{15}$$

where $b_1 = 1.15960$, $b_3 = -0.231919$, etc. The computations of $b_i$ are given in detail in Appendix A. Substituting Eq. (15) into Eq. (14) gives

$$x(t) = A\cos\omega t + \frac{\omega^2 A - b_1 A^{1/3}}{2\omega} t\sin\omega t + \frac{b_3 A^{1/3}}{8\omega^2}(\cos 3\omega t - \cos\omega t). \tag{16}$$

The requirement of no secular terms in $x(t)$ implies that

$$\omega = \frac{\sqrt{b_1}}{A^{1/3}} = \frac{1.07685}{A^{1/3}}. \tag{17}$$

Then,

$$x_H(t) = x(t) = A\cos\omega t - \frac{0.02899 A^{1/3}}{\omega^2}(\cos 3\omega t - \cos\omega t). \tag{18}$$

The exact frequency of the periodic motion of Eq. (13) is given by [14]

$$\omega_e = \frac{\sqrt{\pi}\Gamma(1/4)}{2\sqrt{6}\Gamma(3/4)A^{1/3}} = \frac{1.07045}{A^{1/3}}, \tag{19}$$

where $\Gamma(n)$ is the Gamma function [15]. The approximate frequencies obtained by a first-order harmonic balance solution [12] and a second-order harmonic balance solution [13] are

$$\omega_1 = (4/3)^{1/6}/A^{1/3} = \frac{1.04912}{A^{1/3}}, \tag{20}$$

$$\omega_2 = \frac{1}{\left[\frac{3}{4} + \left(\frac{27}{4}\right)\bar{z} + \left(\frac{243}{2}\right)\bar{z}^2\right]^{1/6}} \left(\frac{1+\bar{z}}{A}\right)^{1/3} = \frac{1.06349}{A^{1/3}} \left(\bar{z} \approx -\frac{1}{51}\right), \tag{21}$$

respectively. We can see that Eq. (17) is more accurate than Eqs. (20) and (21). The second-order harmonic balance approximation $x_M(t)$ to the periodic solution of Eq. (13) is [13]

$$x_M(t) = \frac{A}{1+\bar{z}}(\cos\omega t + \bar{z}\cos 3\omega t). \tag{22}$$

The numerical solution $x_{\mathrm{Num}}(t)$ of Eq. (13) obtained by using Runge–Kutta (R–K) method, and the approximate solutions $x_H(t)$ and $x_M(t)$ computed by Eq. (18) and Eq. (22), respectively, are plotted on Fig. 1 for $A = 1, 100$, and $10\,000$. It can be seen from Fig. 1 that the Mickens solution $x_M(t)$ and the present solution $x_H(t)$ are nearly identical to the numerical solutions.

## 5. Conclusions

A method for nonlinear oscillators is presented using the convolution integral. For the Duffing equation it gives the same results as obtained in Refs. [1,9]. When applied to the nonlinear oscillator equation for which the elastic restoring force is $f(x) = -x^{1/3}$, the method leads to the excellent approximate solutions. Obviously, it is also valid for $f(x) = -x^{(2n+1)/(2m+1)}$ $(m, n = 0, 1, 2, \ldots, n < m)$.

The present method provides an iteration scheme. The iteration procedure can be carried on if solutions of higher degree of accuracy are required. For example, we may substitute Eq. (12) or

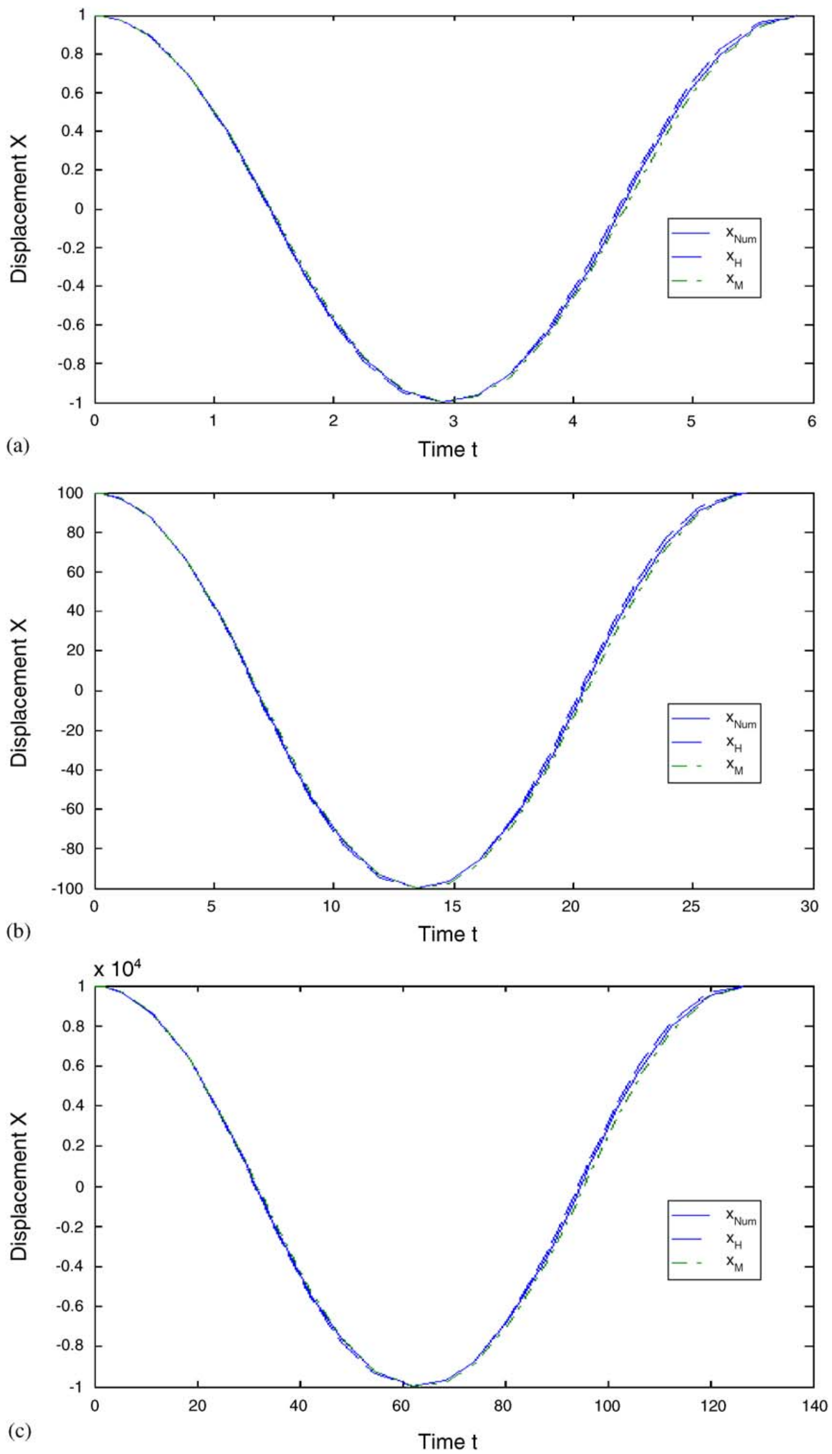


Fig. 1. Comparison of the approximate solutions with the numerical solutions to Eq. (13) for: (a) $A=1$; (b) $A=100$; (c) $A=10\,000$.

Eq. (18) into the right-hand side of Eq. (7) for the second iteration. But when $f(x) = -x^{(2n+1)/(2m+1)}$ ($m,\ n = 0, 1, 2, \ldots, n<m$), the second iteration is not convenient. For the first iteration, Eq. (8) is simple and easy to use. At present, the authors cannot provide a rigorous proof of the convergence of the convolution integral method, for which further research is needed. The possibility of further generalizing the method is now being investigated for the nonlinear oscillating differential equation $\ddot{x} + f(x, \dot{x}) = 0$.

## Acknowledgements

The authors are grateful to the anonymous referees for their helpful comments.

## Appendix A. The computations of $b_i$ for Example 2

Since $(\cos\theta)^{1/3}$ is an even function of $\theta$, the Fourier series expansion of $(\cos\theta)^{1/3}$ is expected to take the form [16]

$$(\cos\theta)^{1/3} = \frac{b_0}{2} + b_1\cos\theta + b_2\cos 2\theta + b_3\cos 3\theta + b_4\cos 4\theta + b_5\cos 5\theta + \cdots, \tag{A.1}$$

where

$$\begin{aligned} b_{2i} &= \frac{2}{\pi}\int_0^{\pi}(\cos\theta)^{1/3}\cos(2i\theta)\,\mathrm{d}\theta \\ &= \frac{2}{\pi}\left[\int_0^{\pi/2}(\cos\theta)^{1/3}\cos(2i\theta)\,\mathrm{d}\theta + \int_{\pi/2}^{\pi}(\cos\theta)^{1/3}\cos(2i\theta)\,\mathrm{d}\theta\right], \quad i = 0, 1, \ldots. \end{aligned} \tag{A.2}$$

Letting $\theta = \pi - \phi$, we have

$$\int_{\pi/2}^{\pi}(\cos\theta)^{1/3}\cos(2i\theta)\,\mathrm{d}\theta = -\int_0^{\pi/2}(\cos\phi)^{1/3}\cos(2i\phi)\,\mathrm{d}\phi, \quad i = 0, 1, \ldots. \tag{A.3}$$

Substituting Eq. (A.3) into Eq. (A.2) gives

$$b_{2i} = 0, \quad i = 0, 1, \ldots. \tag{A.4}$$

Using the relation [17]

$$\int_0^{\pi/2}\cos^n x\,\mathrm{d}x = \int_0^{\pi/2}\sin^n x\,\mathrm{d}x = \frac{\sqrt{\pi}\Gamma(\frac{n}{2}+\frac{1}{2})}{2\Gamma(\frac{n}{2}+1)} \quad (n > -1), \tag{A.5}$$

we have

$$b_1 = \frac{2}{\pi}\int_0^{\pi}(\cos\theta)^{1/3}\cos\theta\,\mathrm{d}\theta = \frac{4}{\pi}\int_0^{\pi/2}(\cos\theta)^{4/3}\,\mathrm{d}\theta = \frac{2\Gamma(7/6)}{\sqrt{\pi}\Gamma(5/3)} = 1.15960, \tag{A.6}$$

$$b_3 = \frac{2}{\pi}\int_0^{\pi} (\cos\theta)^{1/3} \cos 3\theta \, d\theta = \frac{2}{\pi}\int_0^{\pi} (\cos\theta)^{1/3} (4\cos^3\theta - 3\cos\theta) \, d\theta$$
$$= \frac{16}{\pi}\int_0^{\pi/2} (\cos\theta)^{10/3} \, d\theta - \frac{12}{\pi}\int_0^{\pi/2} (\cos\theta)^{4/3} \, d\vartheta = -\frac{2\Gamma(7/6)}{5\sqrt{\pi}\Gamma(5/3)} = -0.231919, \quad (A.7)$$

$$b_5 = \frac{2}{\pi}\int_0^{\pi} (\cos\theta)^{1/3} \cos 5\theta \, d\theta = \frac{2}{\pi}\int_0^{\pi} (\cos\theta)^{1/3} (16\cos^5\theta - 20\cos^3\theta + 5\cos\theta) \, d\theta$$
$$= \frac{4}{\pi}\int_0^{\pi/2} [16(\cos\theta)^{16/3} - 20(\cos\theta)^{10/3} + 5(\cos\theta)^{4/3}] \, d\theta$$
$$= \frac{\Gamma(7/6)}{5\sqrt{\pi}\Gamma(5/3)} = 0.115960. \quad (A.8)$$

## References


[1] R.E. Mickens, *Oscillations in Planar Dynamic Systems*, World Scientific, Singapore, 1996.

[2] A.H. Nayfeh, D.T. Mook, *Nonlinear Oscillations*, Wiley, New York, 1979.

[3] Y.K. Cheung, S.H. Chen, S.L. Lau, A modified Lindstedt–Poincaré method for certain strongly non-linear oscillators, *International Journal of Non-linear Mechanics* 26 (1991) 367–378.

[4] C.W. Lim, B.S. Wu, A modified Mickens procedure for certain non-linear oscillators, *Journal of Sound and Vibration* 257 (2002) 202–206.

[5] J.H. He, A new perturbation technique which is also valid for large parameters, *Journal of Sound and Vibration* 229 (2000) 1257–1263.

[6] L. Meirovitch, *Elements of Vibration Analysis*, McGraw-Hill, New York, 1975.

[7] S.S. Rao,*Mechanical Vibrations*, second ed., Addison-Wesley, Reading, MA, 1990, pp. 184–185.

[8] H. Hu, A modified method of equivalent linearization that works even when the non-linearity is not small, *Journal of Sound and Vibration* 276 (2004) 1145–1149.

[9] W.T. Thomson, *Theory of Vibration with Applications*, second ed., Prentice-Hall, Englewood Cliffs, NJ, 1981.

[10] H. Hu, A classical perturbation technique which is valid for large parameters, *Journal of Sound and Vibration* 269 (2004) 409–412.

[11] H. Hu, Comparison of two Lindstedt–Poincaré type perturbation methods, *Journal of Sound and Vibration* 278 (2004) 437–444.

[12] R.E. Mickens, Oscillations in an $x^{4/3}$ potential, *Journal of Sound and Vibration* 246 (2001) 375–378.

[13] R.E. Mickens, Analysis of non-linear oscillators having non-polynomial elastic terms, *Journal of Sound and Vibration* 255 (2002) 789–792.

[14] H.P.W. Gottlieb, Frequencies of oscillators with fractional-power non-linearities, *Journal of Sound and Vibration* 261 (2003) 557–566.

[15] I.N. Bronshtein, K.A. Semendyayev, *Handbook of Mathematics*, Van Nostrand Reinhold, New York, 1985.

[16] W. Kaplan, *Advanced Calculus*, third ed., Addison-Wesley, Reading, MA, 1984.

[17] R.S. Burington, *Handbook of Mathematical Tables and Formulas*, fifth ed., McGraw-Hill, New York, 1973.